Rotational Energy

\[ K_R = \text{Kinetic Energy due to rotation} \]

\[ K_R = \sum \frac{1}{2} m_i r_i^2 \]

\[ = \frac{1}{2} \sum m_i r_i^2 \omega^2 \]

\[ K_R = \frac{1}{2} I \omega^2 \]

\[ I = \sum m_i r_i^2 \]

Moment of Inertia

I: Measures resistance to rotational motion

E.g.: Moment of inertia of 4 masses

\[ \begin{array}{c}
M \\
\rightarrow \quad a \\
\rightarrow \quad a \\
\end{array} \]

\[ \begin{array}{c}
M \\
\downarrow \quad b \\
\downarrow \quad m \\
\end{array} \]
Rotation about $y$-axis:

\[ I_y = Ma^2 + Ma^2 = 2Ma^2 \]

\[ K_R = \frac{1}{2} (2Ma^2)/\omega^2 = Ma^2\omega^2 \]

Rotation about $z$-axis

\[ I_z = Ma^2 + Ma^2 + mb^2 + mb^2 = 2(Ma^2 + mb^2) \]

\[ K_R = \frac{1}{2} 2(Ma^2 + mb^2)/\omega^2 \]

Moment of Inertia of Continuous objects

\[ I = \lim_{\Delta m \to 0} \sum \Delta m_i r_i^2 = \int r^2 dm \]

For objects with constant density $\rho$,

\[ dm = \rho dV \]

\[ I = \int r^2 \rho dV \]
Eg: Moment of inertia of a hoop of mass $M$ and radius $R$ for rotation about an axis perpendicular to plane of hoop

$$I = \int r^2 \, dm$$
$$= R^2 \int dm$$
$$= R^2 M$$

Eg: Rigid rod of length $L$ and mass $M$ for rotation about center perpendicular to rod

$$\lambda = \frac{M}{L}$$

$$I = \int_{-L/2}^{L/2} x^2 \, dx = \frac{1}{3} x^3 \bigg|_{-L/2}^{L/2} = \frac{1}{12} L^3$$

$$= \frac{M}{L} \frac{1}{12} L^3 = \frac{1}{12} ML^2$$

$$I = \frac{1}{12} ML^2$$

Eg: $I_{\text{sphere}}$ about CM $= \frac{2}{5} MR^2$
Parallel Axis Theorem

Moment of Inertia for rotation about an axis at a point separated by distance $D$ from the CM and an axis parallel to original is

$$I = I_{CM} + MD^2$$

Eg:

Uniform rod about its end:

$$I = I_{CM} + M(L/2)^2$$

$$= \frac{ML^2}{12} + \frac{ML^2}{4}$$

$$= \frac{ML^2}{3}$$
TORQUE: - Measures tendency of a force to rotate an object about an axis.

Eg:

\[ \tau = (FS \sin \phi) r \]

\[ r \sin \phi = d \leftarrow \text{Moment arm (Lever arm)} \]

\[ \tau = Fd \]

Direction of Torque:

\[ \tau = F_1d_1 - F_2d_2 \]

\( \tau \) +ve for counterclockwise
\( \tau \) -ve for clockwise
Newton's 2nd law applied to Rotations

$$\tau = I \alpha$$

[Compare to $F = ma$]

Point particle rotating in a circle:

Torque due to $F_r = 0$ [Fr passes through the axis $\Rightarrow$ Moment arm = 0 for $F_r$]

$$F_t = ma_t \quad a_t = r \alpha$$

$$\tau = F_t r = ma_t r = (mr^2/\alpha)$$

$$\tau = I \alpha$$
VECTOR PRODUCT

\[ \vec{A} \times \vec{B} = \vec{C} \quad \text{Cross product} \]

Eg: \[ \vec{C} = \vec{r} \times \vec{F} \]

\[ \vec{A} \times \vec{B} = -\vec{B} \times \vec{A} \]

\[ \vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C} \]

\[ \frac{d}{dt}(\vec{A} \times \vec{B}) = \frac{d\vec{A}}{dt} \times \vec{B} + \vec{A} \times \frac{d\vec{B}}{dt} \]

\[ c = |\vec{C}| = |\vec{A}| \cdot |\vec{B}| \cdot \sin \theta \]

\[ \theta = \text{Angle between } \vec{A} \text{ and } \vec{B} \]

\[ \hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0 \]

\[ \hat{i} \times \hat{j} = \hat{k} \quad \hat{j} \times \hat{k} = \hat{i} \quad \hat{k} \times \hat{i} = \hat{j} \]

\[ \vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \]

\[ = \hat{i} [A_y B_z - A_z B_y] - \hat{j} [A_x B_z - A_z B_x] + \hat{k} [A_x B_y - A_y B_x] \]
For extended objects:

\[ d\mathbf{F}_t = (dm) \mathbf{a}_t \]

\[ d\tau = \gamma d\mathbf{F}_t = rdm \mathbf{a}_t \]

\[ = (rdm) \tau \alpha \]

\[ \tau = \int r^2 dm \alpha \]

\[ = \alpha \int r^2 dm = I \alpha \]

[\( \alpha \) is same for all \( dm \)]

Eg: Rotating rod: Uniform rod attached to frictionless pivot is free to rotate in a vertical plane. Rod is released from rest in the horizontal position. What is the initial angular acceleration and initial linear acceleration at end of rod?
\[ \alpha = \frac{(Mg)(\frac{L}{2})}{I} \]

\[ I = \frac{1}{3} ML^2 \]

\[ \frac{1}{3} ML^2 \alpha = Mg \frac{L}{2} \]

\[ \Rightarrow \quad \alpha = \frac{3}{2} \frac{g}{L} \]

At end of rod, \( r = 0 \) \( \Rightarrow \ a_t = a_t \)

\[ a_t = \frac{3}{2} g \]

At center of rod, \( r = \frac{L}{2} \) \( \Rightarrow \ a_t = \frac{3}{4} g \)

A penny left at the end will fall slower than the rod!
Example:
\[ \vec{A} = 2\hat{i} + 3\hat{j} \quad \vec{B} = -\hat{i} + 2\hat{j} \]
\[ \vec{A} \times \vec{B} = 4\hat{k} + 3\hat{k} = 7\hat{k} \]

**Angular Momentum**

\[ \vec{L} = \vec{r} \times \vec{p} \]
\[ L = r |p| \sin \theta \]

For linear motion, \( \vec{F} \parallel \vec{v} \Rightarrow \vec{L} = 0 \)

Newton's 2nd law in terms of \( \vec{L} \):

\[ \Sigma \vec{F} = \vec{r} \times \Sigma \vec{F} = \vec{r} \times \frac{d\vec{p}}{dt} \]
\[ \frac{d\vec{L}}{dt} = \frac{d\vec{r} \times \vec{p}}{dt} + \vec{r} \times \frac{d\vec{p}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt} \]
\[ \vec{r} \times \vec{p} = 0 \]

\[ \Rightarrow \quad \Sigma \vec{F} = \frac{d\vec{L}}{dt} \]

Net torque equals rate of change of angular momentum.
ANGULAR MOMENTUM CONSERVATION

\[ \sum \vec{\tau}_{\text{ext}} = \frac{d\vec{L}}{dt} \]

If net torque = 0, \( \frac{d\vec{L}}{dt} = 0 \)

\[ \Rightarrow \quad \vec{L} \text{ constant in time} \]

\[ L_i = L_f \]

\[ I_i \omega_i = I_f \omega_f \]

Example: Spinning skater.

As hands are brought closer to body, skater will spin faster. Here I decreases, to conserve \( L \), \( \omega \) must increase.